

Weighted Sato-Tate Vertical Distribution of the Satake Parameter of Maass Forms on $\mathrm{PGL}(N)$

Fan Zhou

April 8, 2013

Abstract

We formulate a conjectured orthogonality relation between the Fourier coefficients of Maass forms on $\mathrm{PGL}(N)$. Based on the work of Goldfeld and Kontorovich for $N=3$, and on our conjecture for $N \geq 4$, we prove a weighted vertical equidistribution theorem (with respect to the generalized Sato-Tate measure) for the Satake parameter of Maass forms at a finite prime. For $N=3$, the rate of convergence for the equidistribution theorem is obtained.

Keywords: Maass form; automorphic form; Kuznetsov trace formula; Sato-Tate measure; Sato-Tate conjecture; Satake parameter; Casselman-Shalika formula; equidistribution; Ramanujan conjecture; orthogonality relation; trace formula.

Mathematics Subject Classification: Primary 11F55; Secondary 11F72; 11F30.

1 Introduction

Let

$$\varphi(z) = \sum_{n=1}^{\infty} a_{\varphi}(n) e^{2\pi i n z}$$

be a holomorphic modular form of weight k for the modular group $\mathrm{SL}(2, \mathbb{Z})$. We assume that φ is a Hecke eigenfunction with normalization $a_{\varphi}(1) = 1$. The Ramanujan conjecture states

$$\left| \frac{a_{\varphi}(p)}{p^{\frac{k-1}{2}}} \right| \leq 2$$

for a prime number p . It was proved by Deligne in [23] as a consequence of his proof of the Weil conjectures. We define a measure on \mathbb{R}

$$d\mu_{\infty}(x) = \begin{cases} \frac{1}{\pi} \sqrt{1 - \frac{x^2}{4}} \, dx, & \text{when } |x| \leq 2, \\ 0, & \text{otherwise,} \end{cases}$$

called the Sato-Tate measure for $\mathrm{GL}(2)$, or the semi-circle measure. The Sato-Tate conjecture is a more refined statement about the statistics of the Hecke eigenvalues, stating that if φ is a non-CM holomorphic modular form of weight $k \geq 2$, then $\frac{a_{\varphi}(p)}{p^{\frac{k-1}{2}}}$ is an equidistributed sequence

as $p \rightarrow \infty$ with respect to the Sato-Tate measure μ_∞ . More precisely the Sato-Tate conjecture predicts that

$$\lim_{T \rightarrow \infty} \frac{\sum_{p \leq T} f\left(\frac{a_\varphi(p)}{p^{\frac{k-1}{2}}}\right)}{\sum_{p \leq T} 1} = \int_{\mathbb{R}} f \, d\mu_\infty$$

for any continuous test function $f : \mathbb{R} \rightarrow \mathbb{R}$. In recent years many cases of this conjecture have been solved in [3] and [4].

Considering this problem from the vertical perspective, we can fix the prime number p and investigate the distribution of $\frac{a_\varphi(p)}{p^{\frac{k-1}{2}}}$ as φ varies over different automorphic forms. In [21], it was proved that $a_\varphi(p)$ is equidistributed with respect to the p -adic Plancherel measure

$$d\mu_p(x) = \frac{p+1}{(p^{1/2} + p^{-1/2})^2 - x^2} d\mu_\infty(x)$$

as φ runs over all Hecke-Maass cusp forms on $\mathrm{GL}(2)$. An effective version of [21] appeared in [16]. From the same perspective of fixing p and varying φ , [11] and [22] proved similar equidistribution theorems for holomorphic modular forms, which also involve the Plancherel measure. Very recently [24] gave a highbrow generalization of [21], [22] et al.

It is understandable that by fixing a prime number p instead of an automorphic form φ we get the p -adic Plancherel measure instead of the Sato-Tate measure. Strikingly, if we give each Hecke eigenvalue $a_\varphi(p)$ a weight

$$\frac{1}{\mathrm{Res}_{s=1} L(s, \varphi \times \tilde{\varphi})} \quad \left(\text{or } \frac{1}{L(1, \varphi, \mathrm{Ad})} \right)$$

and do the same statistics of fixed p and varying φ , the same Sato-Tate measure appears once again, instead of the Plancherel measure. More interestingly, neither the weight $\frac{1}{\mathrm{Res}_{s=1} L(s, \varphi \times \tilde{\varphi})}$ nor the Sato-Tate measure depends on the choice of the prime number p . In [8] it was essentially proved that

$$\lim_{T \rightarrow \infty} \frac{\sum_{\lambda_\varphi \leq T} \frac{f(a_\varphi(p))}{\mathrm{Res}_{s=1} L(s, \varphi \times \tilde{\varphi})}}{\sum_{\lambda_\varphi \leq T} \frac{1}{\mathrm{Res}_{s=1} L(s, \varphi \times \tilde{\varphi})}} = \int_{\mathbb{R}} f \, d\mu_\infty$$

for any continuous test function $f : \mathbb{R} \rightarrow \mathbb{R}$, where φ runs over all Hecke-Maass forms for $\mathrm{SL}(2, \mathbb{Z})$ and λ_φ is the Laplace eigenvalue of φ . Later [14] and [17] proved similar theorems for holomorphic modular forms. The weight $\frac{1}{\mathrm{Res}_{s=1} L(s, \varphi \times \tilde{\varphi})}$ appears naturally in the Petersson and Kuznetsov trace formulae and that is essential to the proofs.

We generalize theorems of such type to a family of cuspidal automorphic representations of $\mathrm{PGL}(N, \mathbb{A})$. The theory of Maass forms for $\mathrm{SL}(N, \mathbb{Z})$ ($N \geq 3$) has been studied since the 1980s. The definitions and results are summarized in [12]. The cuspidal part of $\mathcal{L}^2(\mathrm{SL}(N, \mathbb{Z}) \backslash \mathrm{GL}(N, \mathbb{R})/\mathrm{O}(N, \mathbb{R}) \cdot \mathbb{R}^\times)$ has a discrete spectrum ϕ_1, ϕ_2, \dots with $0 < \lambda_1 \leq \lambda_2 \leq \dots$ and $\Delta \phi_j = \lambda_j \phi_j$, where Δ is the Laplace operator and each ϕ_j is a Hecke eigenfunction. After adelic lifting, each ϕ_j corresponds to an irreducible un-ramified automorphic representation π_j of $\mathrm{PGL}(N, \mathbb{A})$. The global representation π_j factorizes into local representations $\otimes_{v \leq \infty} \pi_{j,v}$. Each Maass form ϕ_j has

a spectral parameter $\nu^{(j)} = (\nu_1^{(j)}, \dots, \nu_{N-1}^{(j)}) \in \mathbb{C}^{N-1}$, which determines $\pi_{j,\infty}$. Each ϕ_j has Fourier coefficients $A_j(m_1, \dots, m_{N-1})$ for integers m_1, \dots, m_{N-1} with normalization $A_j(1, \dots, 1) = 1$.

For a finite prime p , we have $\pi_{j,p}$ an un-ramified principal series representation of $\mathrm{PGL}(N, \mathbb{Q}_p)$. Denote the standard maximal torus of $\mathrm{SL}(N, \mathbb{C})$ by T , the Weyl group by W , and the standard maximal torus of $\mathrm{SU}(N) \subset \mathrm{SL}(N, \mathbb{C})$ by T_0 . The Satake isomorphism sends each $\pi_{j,p}$ to a point $X_j(p)$ in T/W , which is called the Satake parameter of $\pi_{j,p}$. The generalized Ramanujan conjecture predicts that $\pi_{j,p}$ is tempered and, equivalently, $X_j(p)$ lies in T_0/W , which is a proper subset of T/W . We define the generalized Sato-Tate measure on T_0/W by pushforwarding the normalized Haar measure of $\mathrm{SU}(N)$ to T_0/W that sends an element to its conjugacy class. Denote the Sato-Tate measure on T_0/W by dx . Whereas in $\mathrm{GL}(2)$ the Hecke eigenvalue at p is enough to characterize the local factor at p , it is false when we move to higher dimensions. We shall investigate the distribution of the Satake parameters $X_j(p) \in T/W$ instead of the Hecke eigenvalues, as in [24].

Inspired by previous work on the Kuznetsov trace formula and the Petersson trace formula such as [5], [8], [13], [14], [17], [18], it is natural to formulate the following conjecture.

Conjecture 1.1 (Orthogonality relation). *For each $j = 1, 2, \dots$, let $A_j(m_1, \dots, m_{N-1})$ denote the $(m_1, \dots, m_{N-1})^{\mathrm{th}}$ Fourier coefficient of a Maass form ϕ_j for $\mathrm{SL}(N, \mathbb{Z})$ with $N \geq 2$. For each $T \gg 1$, and each $j = 1, 2, \dots$, let $\omega_j(T)$ be a non-negative real number (weight) satisfying*

$$\sum_{j=1}^{\infty} \omega_j(T) \ll_T 1, \quad \sum_{j=1}^{\infty} A_j(m_1, \dots, m_{N-1}) \overline{A_j(n_1, \dots, n_{N-1})} \omega_j(T) \ll_T 1$$

for all positive integers m_i and n_i . We conjecture that for a proper choice of $\omega_j(T)$ the following orthogonality relation holds:

$$\lim_{T \rightarrow \infty} \frac{\sum_{j=1}^{\infty} A_j(m_1, \dots, m_{N-1}) \overline{A_j(n_1, \dots, n_{N-1})} \omega_j(T)}{\sum_{j=1}^{\infty} \omega_j(T)} = \begin{cases} 1, & \text{if } m_i = n_i \text{ for all } i, \\ 0, & \text{otherwise.} \end{cases}$$

Conjecture 1.1 was proved for $N = 2$ in [8], and for $N = 3$ in [13] and [5], where the following stronger result (with error term) was obtained. Goldfeld and Kontorovich's orthogonality relation states

$$\frac{\sum_{j=1}^{\infty} A_j(m_1, m_2) \overline{A_j(n_1, n_2)} \frac{h_T(\nu^{(j)})}{\mathrm{Res}_{s=1} L(s, \phi_j \times \bar{\phi}_j)}}{\sum_{j=1}^{\infty} \frac{h_T(\nu^{(j)})}{\mathrm{Res}_{s=1} L(s, \phi_j \times \bar{\phi}_j)}} = \delta_{m_1, n_1} \delta_{m_2, n_2} + O_{\{h_T\}, \epsilon}((m_1 m_2 n_1 n_2)^2 T^{\epsilon-2}) \quad (1)$$

for a fixed $\epsilon > 0$ as $T \rightarrow \infty$, where $\{h_T : \mathbb{C}^2 \rightarrow \mathbb{R}\}$ is a family of test functions in which h_T is essentially supported on $\{\phi_j : \lambda_j \leq T^2\}$ (Definition 6.3).

Theorem 1.2 (Main theorem). *Let ϕ_1, ϕ_2, \dots be the basis of Maass forms for $\mathrm{SL}(N, \mathbb{Z})$. Each ϕ_j corresponds to an irreducible un-ramified automorphic representation π_j of $\mathrm{PGL}(N, \mathbb{A})$ with the Satake parameter $X_j(p) \in T/W$ at a finite prime p . Assume Conjecture 1.1 if $N \geq 4$. For any continuous test function $f : T/W \rightarrow \mathbb{C}$, we have the equality*

$$\lim_{T \rightarrow \infty} \frac{\sum_{j=1}^{\infty} f(X_j(p)) \omega_j(T)}{\sum_{j=1}^{\infty} \omega_j(T)} = \int_{T_0/W} f(x) dx. \quad (2)$$

Our main idea of the proof is to translate the Fourier coefficients $A_j(m_1, \dots, m_{N-1})$ in Equation 1 into the characters of finite-dimensional representations of $SU(N)$, via the Casselman-Shalika formula. We complete the proof after some computation and an application of the Peter-Weyl theorem (or the Stone-Weierstrass theorem).

Theorem 1.2 essentially proves that the Ramanujan conjecture $X_j(p) \in T_0/W$ holds in average in the vertical sense, i.e., for fixed p and varying j . This is because the left side of Equation 2 has $X_j(p) \in T/W$ while the Sato-Tate measure dx on the right side of Equation 2 is only supported on T_0/W .

Remark 1.3. After we submitted the preprint of this paper to the Math ArXiv, we were notified that Theorem 1.2 (for the case $N = 3$) was independently proved in [6].

As in [16], [20], and [24], we also obtain an effective version of Theorem 1.2 for $N = 3$, which gives the rate of convergence, but only for monomial functions. Its proof is based on the error term $O_{h,\epsilon}((m_1 m_2 n_1 n_2)^2 T^{\epsilon-2})$ in the orthogonality relation (Equation 1).

Theorem 1.4 (Rate of convergence for $N=3$). *Let ϕ_1, ϕ_2, \dots be the basis of Maass forms for $SL(3, \mathbb{Z})$. Let $f : T/W \rightarrow \mathbb{C}$ be defined as*

$$f\left(\begin{pmatrix} \alpha_1 & & \\ & \alpha_2 & \\ & & \alpha_3 \end{pmatrix}\right) = \left(\sum_{i=1}^3 \alpha_i\right)^{i_1} \left(\sum_{i=1}^3 \overline{\alpha_i}\right)^{i'_1} \left(\sum_{1 \leq i < j \leq 3} \alpha_i \alpha_j\right)^{i_2} \left(\sum_{1 \leq i < j \leq 3} \overline{\alpha_i \alpha_j}\right)^{i'_2}$$

for non-negative integers i_1, i'_1, i_2, i'_2 . For $T \gg 1$, let $\{h_T : \mathbb{C}^2 \rightarrow \mathbb{R}\}$ be a family of test functions where h_T is essentially supported on $\{\phi_j : \lambda_j \leq T^2\}$ as in Definition 6.3. For fixed $\epsilon > 0$, we have the asymptotic formula with error term

$$\frac{\sum_{j=1}^{\infty} f(X_j(p)) \frac{h_T(v^{(j)})}{\text{Res}_{s=1} L(s, \phi_j \times \phi_j)}}{\sum_{j=1}^{\infty} \frac{h_T(v^{(j)})}{\text{Res}_{s=1} L(s, \phi_j \times \phi_j)}} = \int_{T_0/W} f(x) dx + O_{\{h_T\}, \epsilon}((p^2 + 1 + p^{-2})^{i_1+i'_1+i_2+i'_2} T^{\epsilon-2})$$

as $T \gg 1$.

2 Background on Maass Forms

Our main reference is [12] for this section. Fix an integer $N \geq 2$. The cuspidal part of $\mathcal{L}^2(SL(N, \mathbb{Z}) \backslash GL(N, \mathbb{R})/O(N, \mathbb{R}) \cdot \mathbb{R}^\times)$ has a discrete spectrum ϕ_1, ϕ_2, \dots with $\Delta \phi_j = \lambda_j \phi_j$ and $0 < \lambda_1 \leq \lambda_2 \leq \dots$, where Δ is the Laplace operator and each ϕ_j is a Hecke eigenfunction. Via adelic lifting, each Hecke-Maass form ϕ_j corresponds to an irreducible automorphic representation $\pi_j = \otimes_{v \leq \infty} \pi_{j,v}$ of $PGL(N, \mathbb{A})$. The asymptotic behavior of this discrete spectrum, namely, the Weyl law, has been studied since the invention of the Selberg trace formula.

Let \mathfrak{h}^N be the generalized upper half-plane consisting of $z = x \cdot y$, where

$$x = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & x_{1,N} \\ & 1 & x_{2,3} & \cdots & x_{2,N} \\ & & \ddots & & \vdots \\ & & & 1 & x_{N-2,N-1} & x_{N-2,N} \\ & & & & 1 & x_{N-1,N} \\ & & & & & 1 \end{pmatrix} \text{ and } y = \begin{pmatrix} y_1 y_2 \cdots y_{N-1} & & & & \\ & \ddots & & & \\ & & y_1 y_2 & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix}$$

with $x_{*,*} \in \mathbb{R}$ and $y_* > 0$. Each Maass form ϕ_j is a smooth function in $\mathcal{L}^2(\mathrm{SL}(N, \mathbb{Z}) \backslash \mathrm{GL}(N, \mathbb{R})/\mathrm{O}(N, \mathbb{R}) \cdot \mathbb{R}^\times)$. By the Iwasawa decomposition $\mathfrak{h}^N \simeq \mathrm{GL}(N, \mathbb{R})/\mathrm{O}(N, \mathbb{R}) \cdot \mathbb{R}^\times$, we can view ϕ_j as a function on \mathfrak{h}^N invariant on the left by the action of $\mathrm{SL}(N, \mathbb{Z})$. It has Fourier-Whittaker expansion

$$\phi_j(z) = \sum_{\gamma \in \mathrm{U}_{N-1}(\mathbb{Z}) \backslash \mathrm{SL}(N-1, \mathbb{Z})} \sum_{m_1=1}^{\infty} \cdots \sum_{m_{N-2}=1}^{\infty} \sum_{m_{N-1} \neq 0} \frac{A_j(m_1, \dots, m_{N-1})}{\prod_{k=1}^{N-1} |m_k|^{\frac{k(N-k)}{2}}} W_{\mathrm{Jacquet}} \left(M \cdot \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix}, \nu^{(j)} + \frac{1}{N}, \psi_{1, \dots, 1, \frac{m_{N-1}}{|m_{N-1}|}} \right),$$

where

$$M = \begin{pmatrix} m_1 \dots m_{N-2} |m_{N-1}| & & & \\ & \ddots & & \\ & & m_1 m_2 & \\ & & & m_1 \\ & & & & 1 \end{pmatrix}$$

and W_{Jacquet} is Jacquet's Whittaker function. We choose to normalize ϕ_j by requiring that $A_j(1, \dots, 1) = 1$. The $(N-1)$ -tuple $\nu^{(j)} = (\nu_1^{(j)}, \dots, \nu_{N-1}^{(j)}) \in \mathbb{C}^{N-1}$ is the spectral parameter of ϕ_j and it determines the Laplace eigenvalue λ_j and the local principal series representation $\pi_{j, \infty}$ at the infinite place. The Selberg eigenvalue conjecture, or the Ramanujan conjecture at the infinite place, predicts that all $\nu_i^{(j)}$'s are purely imaginary numbers. The number $A_j(m_1, \dots, m_{N-1})$ is the $(m_1, \dots, m_{N-1})^{\mathrm{th}}$ -Fourier coefficient of ϕ_j , which is encoded with critical information about π_j .

3 The Satake Parameter and the Sato-Tate Measure

We give the definitions of the Satake parameters, the Sato-Tate measure and the Ramanujan conjecture for Maass forms on $\mathrm{PGL}(N)$ in this section. We are happy to refer to [24] for more general definitions on other groups.

The standard maximal torus of $\mathrm{SL}(N, \mathbb{C})$ is

$$T = \left\{ \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_N \end{pmatrix} : \alpha_i \in \mathbb{C}^* \text{ for all } i, \prod_{i=1}^N \alpha_i = 1 \right\}.$$

The group $\mathrm{SU}(N)$ is the standard maximal compact subgroup of $\mathrm{SL}(N, \mathbb{C})$. The standard maximal torus of $\mathrm{SU}(N)$ is

$$T_0 = \left\{ \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_N \end{pmatrix} : \alpha_i \in \mathbb{C}^* \text{ and } |\alpha_i| = 1 \text{ for all } i, \prod_{i=1}^N \alpha_i = 1 \right\}.$$

The Weyl group W is isomorphic to the symmetric group of N elements and acts on T and T_0 by permutation of the diagonal entries. The conjugacy classes of $\mathrm{SU}(N)$ (or $\mathrm{SL}(N, \mathbb{C})$) are one-to-one corresponding to elements in T_0/W (or T/W). The space T_0/W has a natural normalized measure. This measure is the pushforward measure of the normalized Haar measure on $\mathrm{SU}(N)$ by the map $\mathrm{SU}(N) \rightarrow T_0/W$ sending an element to its conjugacy class. Let us denote this

measure on T_0/W by dx and we call this measure dx on T_0/W the generalized **Sato-Tate measure**.

For each Hecke-Maass form ϕ_j , adelic lifting gives a global automorphic representation $\pi_j = \otimes_{v \leq \infty} \pi_{j,v}$ of $\mathrm{PGL}(N, \mathbb{A})$. For a finite prime p , each $\pi_{j,p}$ is an un-ramified principal series representation of $\mathrm{PGL}(N, \mathbb{Q}_p)$. By the Satake isomorphism, this un-ramified principal series representation $\pi_{j,p}$ is associated with N nonzero complex numbers $\alpha_{p,1}, \alpha_{p,2}, \dots, \alpha_{p,N}$ with $\prod_{i=1}^N \alpha_{p,i} = 1$. These numbers $\alpha_{p,i}$ determines the representation $\pi_{j,p}$. We can recover $\pi_{j,p}$ by constructing the space

$$\left\{ \text{smooth function } f : \mathrm{PGL}(N, \mathbb{Q}_p) \rightarrow \mathbb{C} : f \left(\begin{pmatrix} t_1 & & * \\ & \ddots & \\ & & t_N \end{pmatrix} g \right) = \left(\prod_{i=1}^N |t_i|^{\frac{N+1}{2}-i} \alpha_{p,i}^{\mathrm{ord}_p(t_i)} \right) f(g) \right\}$$

and $\mathrm{PGL}(N, \mathbb{Q}_p)$ acts on f from the right. The Satake isomorphism sends $\pi_{j,p}$ to a point

$$X_j(p) = \begin{pmatrix} \alpha_{p,1} & & \\ & \ddots & \\ & & \alpha_{p,N} \end{pmatrix} \in T/W.$$

We define this point $X_j(p)$ in T/W as the **Satake parameter** of this un-ramified principal series representation $\pi_{j,p}$ of $\mathrm{PGL}(N, \mathbb{Q}_p)$.

The generalized Ramanujan conjecture claims that $\pi_{j,p}$ is tempered when it comes from a Hecke-Maass form ϕ_j and, equivalently, the Satake parameter $X_j(p)$ lies in T_0/W which is a proper subspace of T/W . More explicitly the Ramanujan conjecture claims that $|\alpha_{p,i}| = 1$ for $i = 1, 2, \dots, N$. The Ramanujan conjecture has not been proved for Maass forms, even when $N = 2$, as of February 2013.

4 The Root System of Type A

The Lie group $\mathrm{SL}(N, \mathbb{C})$ and its maximal compact subgroup $\mathrm{SU}(N)$ are associated with the root system of type A_{N-1} . We construct the A_{N-1} root system in

$$\left\{ (x_1, x_2, \dots, x_N) \in \mathbb{R}^N : \sum_{i=1}^N x_i = 0 \right\}.$$

Let ϵ_i be the vector in \mathbb{R}^N with 1 at the i^{th} entry and 0 elsewhere. We have the set of roots $\Phi = \{\epsilon_i - \epsilon_j : i \neq j\}$. We pick up the set of positive roots $\Phi^+ = \{\epsilon_i - \epsilon_j : i < j\}$. We have $(\epsilon_i - \epsilon_{i+1})$ for $i = 1, 2, \dots, N-1$ as the simple roots of Φ^+ . Denote the zero weight by $\mathbf{0} = (0, \dots, 0)$.

Let Λ be the set of integral weights, which is \mathbb{Z} -module generated by $\left(\epsilon_i - \frac{1}{N} \sum_{j=1}^N \epsilon_j \right)$ for $i = 1, 2, \dots, N-1$. Let $C \subset \left\{ (x_1, x_2, \dots, x_N) \in \mathbb{R}^N : \sum_{i=1}^N x_i = 0 \right\}$ be the Weyl chamber associated with the positive roots in Φ^+ . Explicitly we have

$$C = \left\{ \sum_{i=1}^N a_i \epsilon_i : a_1 \geq a_2 \geq \dots \geq a_N, a_i \in \mathbb{R}, \sum_{i=1}^N a_i = 0 \right\}.$$

For a weight $\mu \in \Lambda \cap C$ we define V_μ as the highest weight representation of μ . It can be a representation of $SU(N)$ or $SL(N, \mathbb{C})$, by the basic Lie theory. Moreover, each irreducible finite-dimensional complex linear representation of $SU(N)$ or $SL(N, \mathbb{C})$ is associated with such a highest weight in $\Lambda \cap C$. Let χ_μ be the character of this representation. The character χ_μ is a well-defined function on conjugacy classes, T/W and T_0/W . Formally each χ_μ is a finite sum of e^η for $\eta \in \Lambda$ with non-negative integer coefficients, invariant under the action of the Weyl group W .

Let V_1 be the representation of the standard defining map $SL(N, \mathbb{C}) \hookrightarrow GL(N, \mathbb{C})$. This representation corresponds to the highest weight representation of $(\epsilon_1 - \frac{1}{N} \sum_{j=1}^N \epsilon_j)$. Its character is

$$\chi_{\epsilon_1 - \frac{1}{N} \sum_{j=1}^N \epsilon_j} \left(\begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_N \end{pmatrix} \right) = \sum_{i=1}^N \alpha_i.$$

Formally we have

$$\chi_{\epsilon_1 - \frac{1}{N} \sum_{j=1}^N \epsilon_j} = \sum_{i=1}^N e^{\epsilon_i - \frac{1}{N} \sum_{j=1}^N \epsilon_j},$$

where $e^{\epsilon_i - \frac{1}{N} \sum_{j=1}^N \epsilon_j}$ corresponds to a character of T or T_0

$$\begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_N \end{pmatrix} \mapsto \alpha_i.$$

Denote the exterior product $\wedge^k V_1$ by V_k for $k = 2, \dots, N-1$ and V_k corresponds to the highest weight representation of $\sum_{i=1}^k \left(\epsilon_i - \frac{1}{N} \sum_{j=1}^N \epsilon_j \right)$. Denote its character $\chi_{\sum_{i=1}^k \left(\epsilon_i - \frac{1}{N} \sum_{j=1}^N \epsilon_j \right)}$ by χ_k for abbreviation. We have the explicit formula

$$\chi_k \left(\begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_N \end{pmatrix} \right) = \chi_{\sum_{i=1}^k \left(\epsilon_i - \frac{1}{N} \sum_{j=1}^N \epsilon_j \right)} \left(\begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_N \end{pmatrix} \right) = \sum_{i_1 < i_2 < \dots < i_k} \alpha_{i_1} \alpha_{i_2} \dots \alpha_{i_k}.$$

These $(N-1)$ representations V_1, \dots, V_{N-1} are the fundamental representations of $SU(N)$ and $SL(N, \mathbb{C})$. It is obvious that $\chi_1, \chi_2, \dots, \chi_{N-1}$ are elementary symmetric polynomials on T/W and T_0/W .

5 The Casselman-Shalika Formula and the Fourier Coefficients at p

Let Ω_N be defined as $\{(l_1, \dots, l_{N-1}) \in \mathbb{Z}^{N-1} : l_1, \dots, l_{N-1} \geq 0\}$. We define a bijective map

$$\aleph : \Omega_N \rightarrow \Lambda \cap C$$

by taking

$$(l_1, \dots, l_{N-1}) \mapsto \sum_{i=1}^{N-1} \left(\sum_{k=1}^{N-i} l_k \right) \left(\epsilon_i - \frac{1}{N} \sum_{j=1}^N \epsilon_j \right).$$

Proposition 5.1 (Casselman-Shalika). *Let ϕ_j be a Hecke-Maass form for $SL(N, \mathbb{Z})$ with Fourier coefficients $A_j(\cdot, \dots, \cdot)$. Let $X_j(p)$ be its Satake parameter at a finite prime p . We have*

$$A_j(p^{l_1}, \dots, p^{l_{N-1}}) = \chi_{\mathbf{s}((l_1, \dots, l_{N-1}))} (X_j(p))$$

for $l_1, \dots, l_{N-1} \geq 0$.

Proof. The Hecke-Maass form ϕ_j can be adelically lifted to a cuspidal automorphic form Φ_j in $\mathcal{L}_{\text{cusp}}^2(\mathbb{Z}(\mathbb{A})\text{GL}(N, \mathbb{Q}) \setminus \text{GL}(N, \mathbb{A}))$. This automorphic form Φ_j has a unique global Whittaker function $W(*; \Phi_j)$. It has factorization

$$W(g; \Phi_j) = \prod_{v \leq \infty} W_v(g_v; \Phi_j).$$

The automorphic form Φ_j generates an automorphic representation π_j of $\text{PGL}(N, \mathbb{A})$ which factorizes into $\otimes_{v \leq \infty} \pi_{j,v}$. With some minor adelic computation, we obtain

$$W_p \left(\begin{pmatrix} p^{l_1 + \dots + l_{N-1}} & & \\ & \ddots & \\ & & p^{l_1} \\ & & & 1 \end{pmatrix}; \Phi_j \right) = \frac{A_j(p^{l_1}, \dots, p^{l_{N-1}})}{\prod_{k=1}^{N-1} p^{\frac{l_k k(N-k)}{2}}}.$$

The un-ramified principal series $\pi_{j,p}$ of $\text{PGL}(N, \mathbb{Q}_p)$ also has a Whittaker function $W_p(*; \pi_{j,p})$ and by normalization $W_p(1; \pi_{j,p}) = 1$ we have

$$W_p \left(\begin{pmatrix} p^{l_1 + \dots + l_{N-1}} & & \\ & \ddots & \\ & & p^{l_1} \\ & & & 1 \end{pmatrix}; \pi_{j,p} \right) = \frac{\chi_{\mathbf{s}((l_1, \dots, l_{N-1}))} (X_j(p))}{\prod_{k=1}^{N-1} p^{\frac{l_k k(N-k)}{2}}}$$

from [10]. By the multiplicity one theorem, we have

$$W_p(*; \pi_{j,p}) = W_p(*; \Phi_j).$$

Evaluating the previous equality at $\begin{pmatrix} p^{l_1 + \dots + l_{N-1}} & & \\ & \ddots & \\ & & p^{l_1} \\ & & & 1 \end{pmatrix}$ we prove the theorem. □

6 The Orthogonality Relation

Recall that ϕ_1, ϕ_2, \dots are Hecke-Maass forms for $SL(N, \mathbb{Z})$ with increasing Laplace eigenvalues. For $j = 1, 2, \dots$, let $\omega_j(T)$ be a non-negative weight associated with ϕ_j and $T \gg 1$ as in Conjecture 1.1. We rewrite Conjecture 1.1 as Conjecture 6.1 and also introduce Conjecture 6.2.

Conjecture 6.1 (Orthogonality relation). *For each $T \gg 1$, and each $j = 1, 2, \dots$, let $\omega_j(T)$ be a non-negative real number (weight) satisfying*

$$\sum_{j=1}^{\infty} \omega_j(T) \ll_T 1, \quad \sum_{j=1}^{\infty} A_j(m_1, \dots, m_{N-1}) \overline{A_j(n_1, \dots, n_{N-1})} \omega_j(T) \ll_T 1$$

for all positive integers m_i and n_i . We conjecture that for a proper choice of $\omega_j(T)$ the following orthogonality relation holds:

$$\lim_{T \rightarrow \infty} \frac{\sum_{j=1}^{\infty} A_j(m_1, \dots, m_{N-1}) \overline{A_j(n_1, \dots, n_{N-1})} \omega_j(T)}{\sum_{j=1}^{\infty} \omega_j(T)} = \begin{cases} 1, & \text{if } m_i = n_i \text{ for all } i, \\ 0, & \text{otherwise.} \end{cases}$$

We predict that Conjecture 6.1 can be derived from the Kuznetsov trace formula with correct choice of a family of test functions and $\omega_j(T)$ should be in the form of

$$\frac{h_T(\nu^{(j)})}{\text{Res}_{s=1} L(s, \phi_j \times \tilde{\phi}_j)},$$

where h_T is a function on the spectral parameters for each $T \gg 1$.

For $N = 2$, Proposition 4.1 of [8] gives a version of Conjecture 6.1. The paper [13] establishes a version of Conjecture 6.1 for $N = 3$. So does [5] afterward. For $N \geq 4$, this conjecture is still open.

Conjecture 6.2 (Weak orthogonality relation). *For each $T \gg 1$, and each $j = 1, 2, \dots$, let $\omega_j(T)$ be a non-negative real number (weight) satisfying*

$$\sum_{j=1}^{\infty} \omega_j(T) \ll_T 1, \quad \sum_{j=1}^{\infty} A_j(m_1, \dots, m_{N-1}) \omega_j(T) \ll_T 1$$

for all integers positive m_i . We conjecture that for a proper choice of $\omega_j(T)$ the following weak orthogonality relation holds:

$$\lim_{T \rightarrow \infty} \frac{\sum_{j=1}^{\infty} A_j(m_1, \dots, m_{N-1}) \omega_j(T)}{\sum_{j=1}^{\infty} \omega_j(T)} = \begin{cases} 1, & \text{if } m_1 = m_2 = \dots = m_{N-1} = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Obviously Conjecture 6.1 implies Conjecture 6.2 because of the normalization $A_j(1, \dots, 1) = 1$. By applying the Casselman-Shalika formula or the Hecke relations, one can prove the inverse is also true. Hence Conjecture 6.1 and Conjecture 6.2 are equivalent.

There are numerous applications of the orthogonality relation. The orthogonality relations with error terms for $N = 2, 3$ are applied to studying the symmetry types of the low-lying zeroes of families of L -functions in [1], [2], [15], and [13]. For $N = 2$, it is also applied to Sato-Tate distribution of Hecke eigenvalues in [8], [17], and [18]. We extend this application further for $N \geq 3$ in Theorem 7.3 and 8.4.

Let us focus on $N = 3$ in the rest of this section.

Definition 6.3. Fix $N = 3$. We define the Rankin-Selberg convolution L -function for a Hecke-Maass form ϕ_j to be

$$L(s, \phi_j \times \tilde{\phi}_j) = \zeta(3s) \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \frac{|A_j(m_1, m_2)|^2}{m_1^{2s} m_2^s},$$

where $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ is the Riemann-Zeta function. We also define

$$\alpha_1^{(j)} = 2\nu_1^{(j)} + \nu_2^{(j)}, \quad \alpha_2^{(j)} = -\nu_1^{(j)} + \nu_2^{(j)}, \quad \alpha_3^{(j)} = -\nu_1^{(j)} - 2\nu_2^{(j)}$$

and

$$\nu_3^{(j)} = \nu_1^{(j)} + \nu_2^{(j)}.$$

For $T \gg 1$ and fixed $R \geq 10$, we define

$$h_{T,R}(\nu^{(j)}) = e^{\frac{(\alpha_1^{(j)2} + \alpha_2^{(j)2} + \alpha_3^{(j)2})}{T^2}} \frac{\left(\prod_{1 \leq i \leq 3} \Gamma\left(\frac{2+R+3\nu_i^{(j)}}{4}\right) \Gamma\left(\frac{2+R-3\nu_i^{(j)}}{4}\right) \right)^2}{\prod_{1 \leq i \leq 3} \Gamma\left(\frac{1+3\nu_i^{(j)}}{2}\right) \Gamma\left(\frac{1-3\nu_i^{(j)}}{2}\right)}.$$

The function $h_{T,R}$ is essentially supported on $\left\{ \phi_j : \lambda_j = 1 - 3 \left(\nu_1^{(j)2} + \nu_2^{(j)2} + \nu_1^{(j)} \nu_2^{(j)} \right) \leq T^2 \right\}$.

Theorem 6.4 (Goldfeld-Kontorovich orthogonality relation). *Let $N = 3$ and assume the Ramanujan conjecture at the infinite place. For $T \gg 1$ and*

$$\omega_j(T) = \frac{h_{T,R}(\nu^{(j)})}{\operatorname{Res}_{s=1} L(s, \phi_j \times \tilde{\phi}_j)},$$

we have

$$\sum_{j=1}^{\infty} A_j(m_1, m_2) \overline{A_j(n_1, n_2)} \omega_j(T) = \begin{cases} \sum_{j=1}^{\infty} \omega_j(T) + O_{R,\epsilon}(T^{3+3R+\epsilon} |m_1 m_2 n_1 n_2|^2), & \text{if } m_1 = n_1 \text{ and } m_2 = n_2, \\ O_{R,\epsilon}(T^{3+3R+\epsilon} |m_1 m_2 n_1 n_2|^2), & \text{otherwise.} \end{cases}$$

Additionally we have the "Weyl law"

$$\sum_j \omega_j(T) \sim c T^{5+3R}$$

for certain $c > 0$.

Proof. See [13]. □

The orthogonality relation for $N = 3$ is generalized to a much larger class of families of test functions in [5].

Theorem 6.5. *Assume $N = 3$ and let m_1, m_2, n_1, n_2 be positive integers. Let $P = m_1 m_2 n_1 n_2$. For $T \gg 1$ and*

$$\omega_j(T) = \frac{h_T(\nu^{(j)})}{\operatorname{Res}_{s=1} L(s, \phi_j \times \tilde{\phi}_j)} \cdot \frac{\left| \prod_{i=1}^3 \Gamma\left(\frac{1}{2} + \frac{3}{2} i \Im \nu_i^{(j)}\right) \right|^2}{\prod_{i,k=1}^3 \Gamma\left(\frac{1+\alpha_i^{(j)} + \overline{\alpha_k^{(j)}}}{2}\right)},$$

where $\theta \leq 7/64$ is a bound towards the Ramanujan conjecture on $GL(2)$. Here h_T is non-negative, uniformly bounded on $\{|\Re v_1| \leq 1/2\} \times \{|\Re v_2| \leq 1/2\}$, $h_T \asymp 1$ on $\{(v_1, v_2) : c \leq \Im v_1, \Im v_2 \leq T, |\Re v_1|, |\Re v_2| \leq 1/2\}$ for some absolute constant $c > 0$, and $h_T(v_1, v_2) \ll_A ((1 + |v_1|/T)(1 + |v_2|/T))^{-A}$. We have

$$\sum_{j=1}^{\infty} A_j(m_1, m_2) \overline{A_j(n_1, n_2)} \omega_j(T) = \begin{cases} \sum_{j=1}^{\infty} \omega_j(T) + O((T^2 P^{1/2} + T^3 P^{\theta} + P^{5/3})(TP)^{\epsilon}), & \text{if } m_1 = n_1 \\ & m_2 = n_2, \\ O((T^2 P^{1/2} + T^3 P^{\theta} + P^{5/3})(TP)^{\epsilon}), & \text{otherwise.} \end{cases}$$

Proof. See the appendix of [5]. □

7 A Short Proof of the Main Theorem Under the Assumption of the Ramanujan Conjecture

Let us assume the Ramanujan conjecture which states that the Satake parameter $X_j(p)$ of a Hecke-Maass form ϕ_j has the property $X_j(p) \in T_0/W$. Let $C(T_0/W)$ be the space of complex-valued continuous functions on T_0/W . It is a Banach space under the supremum norm $\|f\|_{\infty} = \sup_{x \in T_0/W} |f(x)|$. All characters χ_{μ} lie in $C(T_0/W)$. We define the space spanned by characters

$$\mathcal{B} = \left\{ \sum_{\mu \in \Lambda \cap C} a_{\mu} \chi_{\mu} : a_{\mu} \in \mathbb{C}, a_{\mu} = 0 \text{ for all but finitely many } \mu \right\}.$$

Theorem 7.1 (Peter-Weyl). *The space \mathcal{B} is dense in $C(T_0/W)$, under the topology of the supremum norm.*

Proof. See [9] and [7]. This is a less known version of Peter-Weyl theorem than the L^2 -version. □

Lemma 7.2. *Assume Conjecture 6.2 if $N \geq 4$. For any $f \in \mathcal{B}$, we have the equality*

$$\lim_{T \rightarrow \infty} \frac{\sum_j f(X_j(p)) \omega_j(T)}{\sum_j \omega_j(T)} = \int_{T_0/W} f(x) \, dx.$$

Proof. We only need to prove for $f = \chi_{\mu}$ for all $\mu \in \Lambda \cap C$. Recall a corollary of the Schur orthogonality relations

$$\int_{T_0/W} \chi_{\mu}(x) \, dx = \begin{cases} 1, & \text{if } \mu = \mathbf{0}, \\ 0, & \text{otherwise.} \end{cases}$$

If $f = \chi_{\mathbf{0}} \equiv 1$ (constant function), we have

$$\frac{\sum_j f(X_j(p)) \omega_j(T)}{\sum_j \omega_j(T)} = 1 = \int_{T_0/W} f(x) \, dx.$$

If $f = \chi_\mu$ for $\mu \neq \mathbf{0}$, we have

$$\begin{aligned}
\lim_{T \rightarrow \infty} \frac{\sum_j f(X_j(p)) \omega_j(T)}{\sum_j \omega_j(T)} &= \lim_{T \rightarrow \infty} \frac{\sum_j \chi_\mu(X_j(p)) \omega_j(T)}{\sum_j \omega_j(T)} \\
&= \lim_{T \rightarrow \infty} \frac{\sum_j A_j(p^{\aleph^{-1}(\mu)}) \omega_j(T)}{\sum_j \omega_j(T)} \\
&= 0 \\
&= \int_{T_0/W} f(x) \, dx,
\end{aligned}$$

where $A_j(p^{\aleph^{-1}(\mu)})$ means $A_j(p^{l_1}, \dots, p^{l_{N-1}})$ if $\aleph((l_1, \dots, l_{N-1})) = \mu$. Because \aleph is bijective, we have $\aleph^{-1}(\mu) \neq (0, \dots, 0)$ with $\mu \neq \mathbf{0}$ and

$$\lim_{T \rightarrow \infty} \frac{\sum_j A_j(p^{\aleph^{-1}(\mu)}) \omega_j(T)}{\sum_j \omega_j(T)} = 0$$

from Theorem 6.4 ($N = 3$) and Conjecture 6.2 ($N \geq 4$). □

Theorem 7.3 (Main theorem I). *Assume Conjecture 6.2 if $N \geq 4$. Assume the Ramanujan conjecture $X_j(p) \in T_0/W$. For any continuous test function $f \in C(T_0/W)$, we have the equality*

$$\lim_{T \rightarrow \infty} \frac{\sum_{j=1}^{\infty} f(X_j(p)) \omega_j(T)}{\sum_{j=1}^{\infty} \omega_j(T)} = \int_{T_0/W} f(x) \, dx.$$

Proof. We have already proved this theorem when $f \in \mathcal{B}$ and \mathcal{B} is a dense subspace of $C(T_0/W)$. We need a little bit of analysis to complete the proof. For $T \gg 1$, we define a linear functional on $C(T_0/W)$ by

$$\mathbb{L}_T(g) = \frac{\sum_j g(X_j(p)) \omega_j(T)}{\sum_j \omega_j(T)}$$

for $g \in C(T_0/W)$. We define another linear functional by

$$\mathbb{L}_\infty(g) = \int_{T_0/W} g(x) \, dx$$

for $g \in C(T_0/W)$. Both \mathbb{L}_T and \mathbb{L}_∞ are continuous under the supremum norm $\|\cdot\|_\infty$ and we have the inequalities

$$|\mathbb{L}_T(g)| \leq \|g\|_\infty \quad \text{and} \quad |\mathbb{L}_\infty(g)| \leq \|g\|_\infty.$$

By the Peter-Weyl theorem 7.1, any continuous test function f can be approximated under the topology of the supremum norm by functions in \mathcal{B} , i.e., we can find a sequence of functions $f_n \in \mathcal{B}$, $n = 1, 2, \dots$ such that

$$\lim_{n \rightarrow \infty} \|f - f_n\|_\infty = 0.$$

For any $\epsilon > 0$, we can find n' such that $\|f - f_n\|_\infty \leq \frac{\epsilon}{3}$ for any $n > n'$. Since we already have

$$\lim_{T \rightarrow \infty} \mathbb{L}_T(f_{n'+1}) = \mathbb{L}_\infty(f_{n'+1})$$

from the previous lemma. We can find T' such that

$$|\mathbb{L}_T(f_{n'+1}) - \mathbb{L}_\infty(f_{n'+1})| \leq \frac{\epsilon}{3},$$

for any $T > T'$. For any $T > T'$, we have

$$\begin{aligned} |\mathbb{L}_T(f) - \mathbb{L}_\infty(f)| &\leq |\mathbb{L}_T(f) - \mathbb{L}_T(f_{n'+1})| + |\mathbb{L}_T(f_{n'+1}) - \mathbb{L}_\infty(f_{n'+1})| + |\mathbb{L}_\infty(f_{n'+1}) - \mathbb{L}_\infty(f)| \\ &\leq 2\|f - f_{n'+1}\|_\infty + \frac{\epsilon}{3} \\ &\leq \epsilon. \end{aligned}$$

It follows that the limit $\lim_{T \rightarrow \infty} \mathbb{L}_T(f)$ exists and equals $\mathbb{L}_\infty(f)$. □

8 A Long Proof of the Main Theorem Without the Assumption of the Ramanujan Conjecture

In this section, we are going to prove our main theorem without the assumption of the Ramanujan conjecture. Additionally our main theorem will give insight into the Ramanujan conjecture because it will imply a statistical examination of it.

Lemma 8.1. Denote $A_j(1, \dots, \underset{(N-k)^{\text{th}} \text{ position}}{p}, \dots, 1)$ by $A_j[k]$ for abbreviation. We have $A_j[k] = \overline{A_j[N-k]}$ for $k = 1, 2, \dots, N-1$.

Proof. See [12]. □

Let M_p be $p^{\frac{1}{2} - \frac{1}{N^2+1}} > 1$. Denote

$$\left\{ \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_N \end{pmatrix} : \alpha_i \in \mathbb{C}^* \text{ and } |\alpha_i| \leq M_p \text{ for all } i, \prod_{i=1}^N \alpha_i = 1, \right\}$$

by T_1 and we have $T_0 \subset T_1 \subset T$. We shall note that T_1 is a compact set.

Lemma 8.2. The Satake parameter $X_j(p)$ of a Hecke-Maass form ϕ_j lies in T_1/W .

Proof. See [19]. □

We define an injective map $\varrho : T_1/W \rightarrow \mathbb{C}^{N-1}$ by

$$\begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_N \end{pmatrix} \mapsto \left(\chi_1 \left(\begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_N \end{pmatrix} \right), \dots, \chi_{N-1} \left(\begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_N \end{pmatrix} \right) \right).$$

This is a well-defined map because χ_k is invariant under the action of the Weyl group W . Its image $\text{Im } \varrho$ is a compact set in \mathbb{C}^{N-1} . This map establishes the equivalence between $C(T_1/W)$ and $C(\text{Im } \varrho)$ the space of continuous functions on $\text{Im } \varrho$. By the Stone-Weierstrass theorem, polynomials in z_k and $\overline{z_k}$ for $k = 1, 2, \dots, N-1$ on \mathbb{C}^{N-1} are dense in $C(\text{Im } \varrho)$.

Lemma 8.3. Assume Conjecture 6.2 if $N \geq 4$. Let i_k and i'_k be non-negative integers for $k = 1, 2, \dots, N-1$. We have

$$\lim_{T \rightarrow \infty} \frac{\sum_j \prod_{k=1}^{N-1} A_j[k]^{i_k} \overline{A_j[k]}^{i'_k} \omega_j(T)}{\sum_j \omega_j(T)} = \int_{T_0/W} \prod_{k=1}^{N-1} \chi_k(x)^{i_k} \overline{\chi_k(x)}^{i'_k} dx.$$

Proof. By Proposition 5.1, we have $A_j[k] = \chi_k(X_j(p))$. The character of the tensor product representation $\bigotimes_{k=1}^{N-1} (V_k^{\otimes i_k} \otimes V_{N-k}^{\otimes i'_k})$ is $\prod_{k=1}^{N-1} \chi_k^{i_k} \chi_{N-k}^{i'_k}$. By the basic Lie theory, this representation is a direct sum of irreducible representations $\bigotimes_{k=1}^{N-1} (V_k^{\otimes i_k} \otimes V_k^{\otimes i'_k}) = \bigoplus_{\mu \in \Lambda \cap C} V_\mu^{\oplus a_\mu}$, where a_μ is the multiplicity of V_μ . Hence we have the corresponding identity of characters $\prod_{k=1}^{N-1} \chi_k^{i_k} \chi_{N-k}^{i'_k} = \sum_{\mu \in \Lambda \cap C} a_\mu \chi_\mu$. We have

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{\sum_j \prod_{k=1}^{N-1} A_j[k]^{i_k} \overline{A_j[k]}^{i'_k} \omega_j(T)}{\sum_j \omega_j(T)} &= \lim_{T \rightarrow \infty} \frac{\sum_j \prod_{k=1}^{N-1} A_j[k]^{i_k} A_j[N-k]^{i'_k} \omega_j(T)}{\sum_j \omega_j(T)} \\ &= \lim_{T \rightarrow \infty} \frac{\sum_j \prod_{k=1}^{N-1} \chi_k(X_j(p))^{i_k} \chi_{N-k}(X_j(p))^{i'_k} \omega_j(T)}{\sum_j \omega_j(T)} \\ &= \lim_{T \rightarrow \infty} \frac{\sum_j \sum_\mu a_\mu \chi_\mu(X_j(p)) \omega_j(T)}{\sum_j \omega_j(T)} \\ &= \sum_\mu a_\mu \lim_{T \rightarrow \infty} \frac{\sum_j \chi_\mu(X_j(p)) \omega_j(T)}{\sum_j \omega_j(T)} \\ &= a_0. \end{aligned}$$

On the other side, we have

$$\begin{aligned} \int_{T_0/W} \prod_{k=1}^{N-1} \chi_k(x)^{i_k} \overline{\chi_k(x)}^{i'_k} dx &= \int_{T_0/W} \prod_{k=1}^{N-1} \chi_k(x)^{i_k} \chi_k(x)^{i'_k} dx \\ &= \int_{T_0/W} \sum_\mu a_\mu \chi_\mu(x) dx \\ &= \sum_\mu a_\mu \int_{T_0/W} \chi_\mu(x) dx \\ &= a_0. \end{aligned}$$

Hence we establish the equality. □

Theorem 8.4 (Main theorem II). Assume Conjecture 6.2 if $N \geq 4$. For any continuous test function $f : T/W \rightarrow \mathbb{C}$ we have the equality

$$\lim_{T \rightarrow \infty} \frac{\sum_{j=1}^{\infty} f(X_j(p)) \omega_j(T)}{\sum_{j=1}^{\infty} \omega_j(T)} = \int_{T_0/W} f(x) dx. \quad (3)$$

Proof. The composition $f \circ \varrho^{-1}$ is a continuous function in $C(\text{Im } \varrho)$. We only need to prove that for any continuous function $F : \text{Im } \varrho \rightarrow \mathbb{C}$, we have

$$\lim_{T \rightarrow \infty} \frac{\sum_j F(A_j[1], \dots, A_j[N-1]) \omega_j(T)}{\sum_j \omega_j(T)} = \int_{T_0/W} (F \circ \varrho)(x) dx. \quad (4)$$

By the previous lemma, we have proved Equation 4 for F being any monomial $(z_1, \dots, z_{N-1}) \mapsto \prod_{k=1}^{N-1} z_k^{i_k} \overline{z_k}^{i'_k}$. By linear combination, we prove Equation 4 for all polynomials in $z_1, \overline{z_1}, \dots, z_{N-1}, \overline{z_{N-1}}$ on \mathbb{C}^{N-1} . By the Stone-Weierstrass theorem, such polynomials are dense in $C(\text{Im } \varrho)$ under the topology of supremum norm. Apply the same epsilon-delta argument in the proof of Theorem 7.3 and we complete the proof. \square

Remark 8.5. Theorem 8.4 essentially proves the Ramanujan conjecture in average with respect to varying ϕ_j and fixed p . On the left side of Equation 3, $X_j(p)$ is only known to lie in T_1/W , which is the latest record toward proving the Ramanujan conjecture by [19], while on the right side of the same equation, the Sato-Tate measure is supported on exactly T_0/W .

9 Case of $N=3$ and the Rate of Convergence

Fix $N = 3$ in this section and $\omega_j(T)$ is as defined in Theorem 6.4. By an application of the error terms obtained by [13] in Theorem 6.4, we will prove an effective version of Lemma 8.3 for $N = 3$, which estimates the rate of convergence of the limit in that theorem.

Let us recall the formal characters of a representation in the special case of the root system of type A_2 . Characters of $\text{SU}(3)$ or $\text{SL}(3, \mathbb{C})$ are generated by $e^{\mathbf{N}((1,0))} = e^{\epsilon_1 - \frac{\epsilon_1 + \epsilon_2 + \epsilon_3}{3}}$ and $e^{\mathbf{N}((0,1))} = e^{-\epsilon_3 + \frac{\epsilon_1 + \epsilon_2 + \epsilon_3}{3}}$ over \mathbb{Z} as rational functions. We have

$$\chi_1 = e^{\epsilon_1 - \frac{\epsilon_1 + \epsilon_2 + \epsilon_3}{3}} + e^{\epsilon_2 - \frac{\epsilon_1 + \epsilon_2 + \epsilon_3}{3}} + e^{\epsilon_3 - \frac{\epsilon_1 + \epsilon_2 + \epsilon_3}{3}} \quad \text{and} \quad \chi_2 = e^{-\epsilon_1 + \frac{\epsilon_1 + \epsilon_2 + \epsilon_3}{3}} + e^{-\epsilon_2 + \frac{\epsilon_1 + \epsilon_2 + \epsilon_3}{3}} + e^{-\epsilon_3 + \frac{\epsilon_1 + \epsilon_2 + \epsilon_3}{3}}.$$

for the two fundamental representation V_1 and $V_2 = \wedge^2 V_1$.

Theorem 9.1 (Rate of convergence for $N=3$). Fix $N = 3$ and assume the Ramanujan conjecture at the infinite place. Let us fix $\epsilon > 0$ and $R \geq 10$. Keep $\omega_j(T) = \frac{h_{T,R}(v^{(j)})}{\text{Res}_{s=1} L(s, \phi_j \times \tilde{\phi}_j)}$ as defined in

Theorem 6.4. Let $f \circ \varrho$ be a monomial $(z_1, z_2) \mapsto z_1^{i_1} \overline{z_1}^{i'_1} z_2^{i_2} \overline{z_2}^{i'_2}$ for non-negative integers i_1, i'_1, i_2, i'_2 . We have

$$\frac{\sum_{j=1}^{\infty} f(X_j(p)) \omega_j(T)}{\sum_{j=1}^{\infty} \omega_j(T)} = \int_{T_0/W} f(x) dx + O_{R,\epsilon}((p^2 + 1 + p^{-2})^{i_1 + i'_1 + i_2 + i'_2} T^{\epsilon-2})$$

as $T \gg 1$.

Proof. Let us recall $\Omega_3 = \{(l_1, l_2) : l_1, l_2 \in \mathbb{Z}, l_1 \geq 0, l_2 \geq 0\}$ and the map $\aleph : \Omega_3 \rightarrow \Lambda \cap C$ defined by $(l_1, l_2) \mapsto l_1(\epsilon_1 - \frac{\epsilon_1 + \epsilon_2 + \epsilon_3}{3}) + l_2(\frac{\epsilon_1 + \epsilon_2 + \epsilon_3}{3} - \epsilon_3)$. We also recall $V_1^{\otimes(i_1+i'_2)} \otimes V_2^{\otimes(i_2+i'_1)} = \bigoplus_{\mu} V_{\mu}^{a_{\mu}}$ and $\chi_1^{i_1+i'_2} \chi_2^{i_2+i'_1} = \sum_{\mu} a_{\mu} \chi_{\mu}$ in the proof of Lemma 8.3. We have

$$\begin{aligned}
\frac{\sum_j f(X_j(p)) \omega_j(T)}{\sum_j \omega_j(T)} - \int_{T_0/W} f(x) \, dx &= -a_0 + \frac{\sum_j \prod_{k=1}^2 A_j[k]^{i_k} \overline{A_j[k]}^{i'_k} \omega_j(T)}{\sum_j \omega_j(T)} \\
&= -a_0 + \frac{\sum_j \prod_{k=1}^2 A_j[k]^{i_k} A_j[3-k]^{i'_k} \omega_j(T)}{\sum_j \omega_j(T)} \\
&= -a_0 + \frac{\sum_j \chi_1^{i_1+i'_2}(X_j(p)) \chi_2^{i_2+i'_1}(X_j(p)) \omega_j(T)}{\sum_j \omega_j(T)} \\
&= -a_0 + \frac{\sum_j \sum_{\mu} a_{\mu} \chi_{\mu}(X_j(p)) \omega_j(T)}{\sum_j \omega_j(T)} \\
&= -a_0 + \sum_{\mu \in \Lambda \cap C} a_{\mu} \frac{\sum_j \chi_{\mu}(X_j(p)) \omega_j(T)}{\sum_j \omega_j(T)} \\
&= -a_0 + \sum_{l_1 \geq 0, l_2 \geq 0} a_{\aleph((l_1, l_2))} \frac{\sum_j A_j(p^{l_1}, p^{l_2}) \omega_j(T)}{\sum_j \omega_j(T)} \\
&= \sum_{l_1 \geq 0, l_2 \geq 0} a_{\aleph((l_1, l_2))} \left(\frac{\sum_j A_j(p^{l_1}, p^{l_2}) \omega_j(T)}{\sum_j \omega_j(T)} - \delta_{l_1, 0} \delta_{l_2, 0} \right) \\
&= \left(\sum_{l_1 \geq 0, l_2 \geq 0} a_{\aleph((l_1, l_2))} p^{2l_1+2l_2} \right) \mathcal{O}_{R, \epsilon}(T^{\epsilon-2}).
\end{aligned}$$

The last equality comes from the error terms in Theorem 6.4. We need a good bound for $\sum_{l_1 \geq 0, l_2 \geq 0} a_{\aleph((l_1, l_2))} p^{2l_1+2l_2}$. Recall that a_{μ} is the multiplicity of V_{μ} in the decomposition of the representation

$$V_1^{\otimes(i_1+i'_2)} \otimes V_2^{\otimes(i'_1+i_2)} = \bigoplus_{\mu \in \Lambda \cap C} V_{\mu}^{a_{\mu}}.$$

Because of

$$\dim\{v \in V_{\mu} : t.v = \mu(t)v \text{ for all } t \in T_0\} = 1$$

we have

$$\begin{aligned} a_\mu &\leq \dim\{v \in V_1^{\otimes(i_1+i'_2)} \otimes V_2^{\otimes(i'_1+i_2)} : t.v = \mu(t)v \text{ for all } t \in T_0\} \\ &= \chi_1^{i_1+i'_2} \chi_2^{i_2+i'_1} \Big|_{e^\mu}, \end{aligned}$$

where $\Big|_{e^\mu}$ means taking the coefficient before e^μ . Hence we obtain

$$\begin{aligned} \sum_{l_1 \geq 0, l_2 \geq 0} a_{\mathbf{N}((l_1, l_2))} p^{2l_1+2l_2} &\leq \sum_{l_1 \geq 0, l_2 \geq 0} \left(\chi_1^{i_1+i'_2} \chi_2^{i_2+i'_1} \Big|_{e^{\mathbf{N}((l_1, l_2))}} \right) p^{2l_1+2l_2} \\ &= \sum_{\mu \in \Lambda \cap C} \left(\chi_1^{i_1+i'_2} \chi_2^{i_2+i'_1} \Big|_{e^\mu} \right) e^\mu \Big\|_{e^{\mathbf{N}((1,0))}=e^{\mathbf{N}((0,1))}=p^2} \\ &< \sum_{\mu \in \Lambda} \left(\chi_1^{i_1+i'_2} \chi_2^{i_2+i'_1} \Big|_{e^\mu} \right) e^\mu \Big\|_{e^{\mathbf{N}((1,0))}=e^{\mathbf{N}((0,1))}=p^2} \\ &= \chi_1^{i_1+i'_2} \chi_2^{i_2+i'_1} \Big\|_{e^{\mathbf{N}((1,0))}=e^{\mathbf{N}((0,1))}=p^2} \\ &= (p^2 + 1 + p^{-2})^{i_1+i'_1+i_2+i'_2}, \end{aligned}$$

where $\Big\|_{e^{\mathbf{N}((1,0))}=e^{\mathbf{N}((0,1))}=p^2}$ means replacing $e^{\mathbf{N}((1,0))}$ and $e^{\mathbf{N}((0,1))}$ with p^2 in $\chi_1^{i_1+i'_2} \chi_2^{i_2+i'_1}$, which is a rational function generated by $e^{\mathbf{N}((1,0))}$ and $e^{\mathbf{N}((0,1))}$. \square

Acknowledgment

The author is grateful to Dorian Goldfeld, who brought the author to this topic of research, gave the author much guidance, and painstakingly read the manuscript. The author would like to thank Peter Sarnak for many valuable comments, in particular, suggesting using Satake parameters instead of Hecke eigenvalues for equidistribution theorems in higher dimensions. The author would like to thank Hang Xue for helpful discussions.

References

- [1] L. Alpoge and S. J. Miller, *Low-lying zeroes of Maass form L-functions*, preprint.
- [2] L. Alpoge, N. Amersi, G. Iyer, O. Lazarev, S. J. Miller and L. Zhang, *The low-lying zeros of cuspidal Maass forms on $SL(2, \mathbb{Z})$* , preprint.
- [3] T. Barnet-Lamb, T. Gee, and D. Geraghty, *The Sato-Tate conjecture for Hilbert modular forms*, J. Amer. Math. Soc. 24 (2011), no. 2, 411-469.
- [4] T. Barnet-Lamb, D. Geraghty, M. Harris, and R. Taylor, *A family of Calabi- Yau varieties and potential automorphy II*, Publ. Res. Inst. Math. Sci. 47 (2011), no. 1, 29-98.

- [5] V. Blomer, *Applications of the Kuznetsov formula on $GL(3)$* , Invent. Math., to appear.
- [6] V. Blomer, J. Buttcane and N. Raulf, *A Sato-Tate law for $GL(3)$* , Comment. Math. Helv. to appear
- [7] T. Bröcker and T. tom Dieck, *Representations of compact Lie groups*, Graduate Texts in Mathematics 98, Springer-Verlag, New York, 1995.
- [8] R. Bruggeman, *Fourier coefficients of cusp forms*, Invent. Math. 45 (1978), no. 1, 1-18.
- [9] D. Bump, *Lie groups*, Graduate Texts in Mathematics 225, Springer-Verlag, New York, 2004.
- [10] W. Casselman and J. Shalika, *The unramified principal series of p -adic groups. II. The Whittaker function*, Compositio Mathematica, 41 no. 2 (1980), p. 207-231.
- [11] B. Conrey, W. Duke, and D. Farmer, *The distribution of the eigenvalues of Hecke operators*, Acta Arith. 78 (1997), no. 4, 405-409.
- [12] D. Goldfeld, *Automorphic forms and L -functions for the group $GL(n, \mathbb{R})$* , Cambridge Studies in Advanced Mathematics 99, Cambridge University Press, 2006.
- [13] D. Goldfeld and A. Kontorovich, *On the $GL(3)$ Kuznetsov formula with applications to symmetry types of families of L -functions*, in: Automorphic representations and L -functions, Tata Institute of Fundamental Research, Mumbai, India, 2013.
- [14] S. Gun, M. R. Murty and P. Rath, *Summation methods and distribution of eigenvalues of Hecke operators*, Functiones et Approximatio XXXIX.2 (2008), 191-204.
- [15] A. M. Güloğlu, *Low-Lying Zeros of Symmetric Power L -Functions*, Internat. Math. Res. Notices (2005), no. 9, 517-550.
- [16] Y.-K. Lau and Y. Wang, *Quantitative version of the joint distribution of eigenvalues of the Hecke operators*, J. Number Theory 131 (2011), 2262-2281.
- [17] C. Li, *Kuznetsov trace formula and weighted distribution of Hecke eigenvalues*, J. Number Theory 104 (2004), no. 1, 177-192.
- [18] A. Knightly and C. Li, *Kuznetsov's trace formula and the Hecke eigenvalues of Maass forms*, Memoirs of the AMS, to appear.
- [19] W. Luo, Z. Rudnick and P. Sarnak, *On the generalized Ramanujan conjecture for $GL(n)$* , Automorphic forms, automorphic representations, and arithmetic (Fort Worth, TX, 1996), Proc. Sympos. Pure Math. 66, Part 2, Amer. Math. Soc., Providence, RI, 301-10.
- [20] R. Murty and K. Sinha, *Effective equidistribution of eigenvalues of Hecke operators*, J. Number Theory 129 (2009), no. 3, 681-714.
- [21] P. Sarnak, *Statistical properties of eigenvalues of the Hecke operators*, in: Analytic number theory and Diophantine problems (Stillwater, OK, 1984), Progr. Math., 70, Birkhäuser, Boston, MA, 1987, 321-331.

- [22] J.-P. Serre, *Répartition asymptotique des valeurs propres de l'opérateur de Hecke T_p* , J. Amer. Math. Soc. 10 (1997), no. 1, 75-102.
- [23] P. Deligne, *La conjecture de Weil: I*, Publications Mathématiques de l'IHÉS, 43 (1974), p. 273-307.
- [24] S.-W. Shin and N. Templier, *Sato-Tate theorem for families and low-lying zeros of automorphic L -functions*, preprint.